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HYDROGASDYNAMICS IN TECHNOLOGICAL PROCESSES

PRESSURE-RESTORATION CURVE FOR A FRACTAL CRACKED MEDIUM

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The problem on flow of a single-phase, weakly compressible fluid in a cracked medium with the fractal geometry of cracks has been considered. The problem on the pressure-restoration curve has been solved in the approximation of an axisymmetric flow near the well. The corrections to the formula obtained in the classical theory of filtration for the pressure-restoration curve have been computed for the case where the dimension of a system of cracks is close to the dimension of a confining volume.

In modeling fluid flows in porous materials, one usually uses the continuum approximation. It is assumed that the Euclidean-space geometry can be taken as the basis in describing the system's properties, and the macroscopic characteristics of a material (for example, porosity and permeability) and a saturating fluid (for example, density, pressure, and viscosity) can be determined at each point of the space. However, the continuum approximation is inadequate for a number of materials with a cracked porosity and the geometric apparatus of fractal theory should be used [1–3]. A new geometric characteristic of the system — the fractal dimension of a system of cracks d — appears. By definition, this quantity does not exceed the dimension of a confining volume D. Different formulations of the problem on flow in porous materials with the fractal geometry of cracks have been considered in [4, 5].

Of special interest are inverse problems enabling one to determine the parameters of a porous material from the characteristics of flow observed. The widespread method of studying the transport properties of the critical area of formation in developing oil and gas fields is measurement of the pressure-restoration curve. The essence of investigation is a fast shutdown of the producing well operating in the stationary regime, after which the bottom-hole pressure grows, or, in other words, is restored. From the shape of the pressure curve, we can determine the properties of the rock in the critical area both in the classical filtration theory [6] and with the use of complicated filtration models [7, 8]. The problem on the curve of pressure restoration in a cracked-porous medium (i.e., in a medium with two types of porosity) with the fractal geometry of cracks was considered in [9]. In the present work, we have solved the problem on the pressure-restoration curve for cracked rocks with one type of porosity for a low positive value of the dimension defect $\delta = D - d$. As a result we have computed the corrections to the formula for the pressure-restoration curve obtained in the classical filtration theory [6].

Let us consider flow of a weakly compressible fluid in the vicinity of the producing well. We will assume that the flow is cylindrically symmetric in the average sense and isothermal. We write the law of conservation of the fluid mass in the cracks in integral form:

$$\frac{d}{dt}\int_{r_1}^{r_2} m\rho d\mu_{\mathrm{H}}^{d} = \int_{r=r_1} j d\mu_S^d - \int_{r=r_2} j d\mu_S^d . \tag{1}$$

Here r_1 and r_2 are the arbitrary values of the radial coordinate, m is the geometric factor characterizing the opening of the cracks; $d\mu_{\mathrm{H}}^{d}$ is the Hausdorf measure on the system of cracks [10], and $d\mu_S^d$ is the measure at the intersection

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of the fractal and the circle of radius r. These measures are related by the formula $d\mu_H^d = dr d\mu_S^d$. We note that we have $D = 2$ in the formulation in question. Furthermore, the relations

$$\int\limits_{r=a} d\mu_S^d = a^{d-1} \int\limits_{r=1} d\mu_S^d = a^{d-1} s_d , \quad s_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} , \tag{2}$$

where s_d is the area of the unit $(d-1)$-dimensional sphere and $\Gamma(z)$ is the gamma function [11], hold true.

For the fractal system of cracks, it is acceptable to take the expression for the flow based on the ordinary Darcy law [4, 5]:

$$j = -k\rho\mu^{-1} \frac{\partial p}{\partial r} , \tag{3}$$

here k is the analog of permeability for the fractal system of cracks, μ is the shear viscosity of the fluid, $p = p_* + E(\rho - \rho_*)/\rho_*$ is the pressure in the fluid, and ρ_* and p_* are the quiescent (unperturbed) values of the density and the pressure. We introduce $E_s = (\partial m/\partial p)^{-1}$ — the modulus of dilatation of the rock skeleton. The values of μ, E, and E_s will be considered to be constant. Using the assumptions made and Eqs. (1)–(3), we obtain a partial derivative equation determining the dynamics of pressure:

$$\begin{gathered} r^{d-1}\partial_t p = \kappa^2 \partial_r (r^{d-1} \partial_r p) , \\ \kappa^2 = \mu^{-1} (mE^{-1} + E_s^{-1})^{-1} k . \end{gathered} \tag{4}$$

Equation (4) must be solved with specified boundary conditions on the well

$$(s_d k \mu^{-1} r^{d-1} h \partial_r p)\Big|_{r=r_w} = Q , \quad t < 0 ; \tag{5}$$

$$(s_d k \mu^{-1} r^{d-1} h \partial_r p)\Big|_{r=r_w} = 0 , \quad t > 0 \tag{6}$$

and on the supply circuit

$$p\big|_{r=r_c} = p_c . \tag{7}$$

As a result of solution of problem (4)–(7), we determine the bore-hole pressure:

$$p_w = p_w(t) = p\Big|_{r=r_w} . \tag{8}$$

As has been indicated earlier, at $t < 0$, we have filtration stationary flow. From Eqs. (4), (5), and (7), we can find the corresponding pressure distribution:

$$\begin{gathered} p_0(r) = p_c + (2-d)^{-1} q (r^{2-d} - r_c^{2-d}) , \\ q = Q\mu (s_d k h)^{-1} . \end{gathered} \tag{9}$$

It is convenient to pass from the pressure $p(t, r)$ to the function characterizing the change in the pressure in relation to the stationary solution (9):

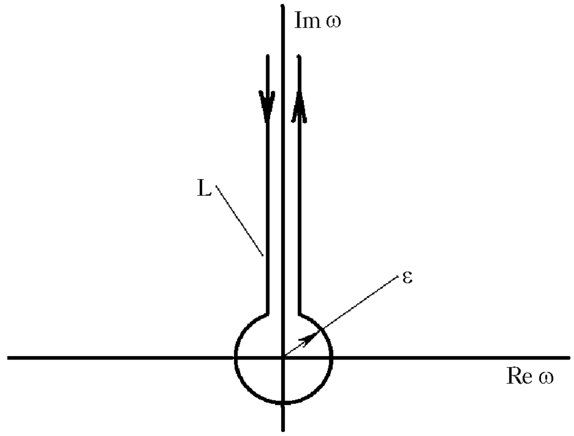

Fig. 1. Pressure-restoration curve for a fractal cracked medium.

$$\Delta p\,(t, r) = p\,(t, r) - p_0\,(r)\,,$$

and perform the Fourier transformation

$$f\,(\omega, r) = \int \exp\,(-\,i\omega t)\,\Delta p\,(t, r)\,dt\,. \tag{10}$$

The problem of determination of the pressure-restoration curve (8) is equivalent to the problem of determination of the inverse Fourier transform of the function (10) on the bore hole:

$$p_{\mathrm{w}}\,(t) - p_0\,(r_{\mathrm{w}}) = \Delta p\,(t, r_{\mathrm{w}}) = (2\pi)^{-1} \int \exp\,(i\omega t)\,f\,(\omega, r_{\mathrm{w}})\,d\omega\,. \tag{11}$$

We perform the Fourier transformation (10) and consider the problem of determination of the function $f(\omega, r)$. Equation (4) and boundary conditions (5) and (6) yield the ordinary differential equation

$$i\omega r^{d-1} f = \kappa^2 \partial_r\,(r^{d-1} \partial_r f) \tag{12}$$

and one boundary condition:

$$\partial_r f\Big|_{r=r_{\mathrm{w}}} = r_{\mathrm{w}}^{1-d}\,iq\,(\omega - i\varepsilon)^{-1}\,, \tag{13}$$

where ε is the infinitesimal positive quantity. Furthermore, since the part of the pressure-restoration curve unaffected by the position of the supply circuit is of practical interest, instead of condition (7) it is convenient to use the condition of decrease in the disturbances of the pressure field at spatial infinity:

$$f\big|_{r\to+\infty} = 0\,. \tag{14}$$

The solution of problem (12)–(14) is expressed by the MacDonald functions [12]:

$$f = -\,r_{\mathrm{w}}^{1-d}\,iq\,((\omega - i\varepsilon)\,z_{\mathrm{w}}^{\nu} K_{1-\nu}\,(z_{\mathrm{w}})\,\alpha)^{-1} z^{\nu} K_{\nu}\,(z)\,, \tag{15}$$

where the function $\alpha = \alpha(\omega)$ is determined from the conditions $\alpha^2 = i\omega/\kappa^2$ and Re $\alpha \geq 0$ and the notation $z = \alpha r$, $z_{\mathrm{w}} = \alpha r_{\mathrm{w}}$, and $\nu = \delta/2 = (2 - d)/2$ is used.

Based on the exact solution (15), we compute the asymptotics of expression (11) at long times. For this purpose, it is convenient to deform the integration contour in (11) to the imaginary axis. A new integration contour L follows the edges of the cut of the integrand and bypasses the point $\omega = 0$ (see Fig. 1). Next, the asymptotics of the pressure-restoration curve at long times is determined by the asymptotics of the integrand at small ω. To compute the leading terms of the asymptotics of the integrand

$$\varphi = \varphi(\omega) = f(\omega, r_w) = r_w^{1-d} q \left(i\omega K_{1-\nu}(z_w)\, \alpha\right)^{-1} K_\nu(z_w) \tag{16}$$

we use the definitions of the Macdonald functions [12]:

$$K_\nu(z) = \frac{\pi}{2\sin(\nu\pi)} \left(I_{-\nu}(z) - I_\nu(z)\right),$$

$$I_\nu(z) = \sum_{m=0}^{+\infty} \left(2^{2m+\nu}\, m!\, \Gamma(m+\nu+1)\right)^{-1} z^{2m+\nu}.$$

Using them, we easily compute the leading terms in the expansion of the function (16) for small ω:

$$\begin{gathered} \varphi = \varphi(\omega) = (i\omega)^{-1} \left(A_d (i\omega)^{-\nu} - B_d + O\left(|\omega|^{d-1}\right)\right), \\ A_d = 2^{1-d} \kappa^{2-d} q \Gamma(1+\nu) \left(\nu \Gamma(1-\nu)\right)^{-1}, \quad B_d = 2^{-1} q \nu^{-1} r_w^{2-d}. \end{gathered} \tag{17}$$

Substituting expression (17) into formula (11) and allowing for the form of the integration contour (see Fig. 1), we obtain

$$\begin{gathered} \Delta p(t, r_w) = J_{1\varepsilon} + J_{2\varepsilon} - B_d + O\left(t^{1-d}\right), \\ J_{1\varepsilon} = A_d \varepsilon^{-\nu} (\nu\pi)^{-1} \sin(\nu\pi), \quad J_{2\varepsilon} = -\pi^{-1} A_d \sin(\nu\pi) \int_\varepsilon^{+\infty} \exp(-ty)\, y^{-1-\nu}\, dy. \end{gathered} \tag{18}$$

Let us compute the limit of the right-hand side of relation (18) for $\varepsilon \to 0$. For this purpose, we use the tabulated expressions [13, formulas 8.350.2 and 8.354.2]

$$\Gamma(-\nu, \varepsilon) = \int_\varepsilon^{+\infty} \exp(-y)\, y^{-\nu-1}\, dy,$$

$$\Gamma(-\nu, \varepsilon) = \Gamma(-\nu) - \sum_{n=0}^{+\infty} \frac{(-1)^n \varepsilon^{-\nu+n}}{n!\,(-\nu+n)},$$

using which we directly pass to the limit $\varepsilon \to 0$. As a result, we obtain the asymptotics of the pressure-restoration curve for the fractal cracked medium:

$$\Delta p(t, r_w) = (2\nu)^{-1} q r_w^{2-d} \left((\nu\pi)^{-1} \Gamma(1+\nu) \sin(\nu\pi) \left(4\kappa t r_w^{-2}\right)^\nu - 1\right) + O\left(t^{1-d}\right). \tag{19}$$

This formula holds true for any values of the dimension d of the system of cracks in the interval $1 < d < 2$. Of special interest is the case where the dimension d of the system of cracks is close to the dimension of the confining space $D = 2$, i.e., where the parameter ν is small. By expansion in the small parameter ν, from formula (19) we can derive the pressure-restoration curve for the classical filtration theory (limit $\nu = 0$) and the correction related to the fractal geometry of cracks. It is necessary to take account of the formula for the Γ function [13, 8.321]

$$\Gamma(1+\nu) = 1 - C\nu + O(\nu^2),$$

where $C \approx 0.577$ is the Euler constant. If the terms having order two or higher in relation to ν are dropped, the formula for the pressure-restoration curve acquires a nearly classical form:

$$\Delta p(t, r_w) \approx 2^{-1} q\,(1 + (2^{-1} - C)\,\nu \ln(4\kappa t r_w^{-2}))\ln(4\kappa t r_w^{-2}) + O(1). \quad (20)$$

It should be noted that it is only the terms growing with time that are written explicitly in expression (20). The range of application of the asymptotics (20) is much narrower than that of (19). In actual fact, both the smallness of the parameter ν and the smallness of the quantity $\nu \ln(4\kappa t r_w^{-2})$ are necessary. Nonetheless, such a form of representation of the pressure-restoration curve can be used in processing experimental data to reveal the distorting influence of the fractal structure of cracks.

Thus, expressions (19) and (20) solve the problem posed in the present work. We certainly know of many other factors capable of causing deviations from the classical form of a pressure-restoration curve in the practice of interpretation of hydrodynamic investigations [6–8]. The asymptotics (19) and (20) found can be used as additional "patterns" in making decisions on the filtration properties of the critical area of formation.

NOTATION

a and A_d, auxiliary quantities, m and kg/(m·sec$^{2+\nu}$); B_d, $J_{1\varepsilon}$, and $J_{2\varepsilon}$, auxiliary quantities, kg/(m·sec^2); C, Euler constant; d, fractal dimension of the system of cracks; D, dimension of the confining space; $d\mu_H^d$, Hausdorff measure on the system of cracks, m^d; $d\mu_S^d$, measure at intersection of the fractal and the circle, m$^{(d-1)}$; E, modulus of dilatation of the fluid, kg/(sec^2·m); E_s, modulus of dilatation of the rock skeleton, kg/(sec^2·m); f, auxiliary function, kg/(sec·m); j, radial fluid flow, kg/(sec·m^d); h, thickness of the productive bed, m; k, permeability for the fractal system of cracks, m$^{(4-d)}$; K_ν and I_ν, Macdonald functions; m, parameter characterizing the opening of cracks, m$^{(2-d)}$; p, pressure in the fluid, kg/(sec^2·m); p_c, supply-circuit pressure, kg/(sec^2·m); p_w, bore-hole pressure, kg/(sec^2·m); p_0, initial pressure distribution, kg/(sec^2·m); q, auxiliary quantity, kg/(m$^{(3-d)}$·sec^2); Q, volume flow rate of the well, m^3/sec; r, radial coordinate; r_w, bit radius of the well, m; r_c, radius of the supply circuit of the well, m; t, time, sec; z and z_w, auxiliary complex quantities; α, auxiliary complex quantity, m^{-1}; δ, dimension defect; ε, infinitesimal positive quantity; $\Gamma(x)$, gamma function; $\Gamma(-\nu, \varepsilon)$, incomplete gamma function; φ, auxiliary function, kg/(sec·m); κ, auxiliary quantity, m/sec$^{1/2}$; μ, shear viscosity of the fluid, kg/(sec·m); ν, auxiliary quantity; ρ, mass density of the fluid, kg/m^3; ω, frequency, sec^{-1}. Subscripts: c, value of the quantity on the supply circuit; s, parameter of the rock skeleton; S, surface; w, value of the quantity in the well; *, unperturbed state of the fluid.

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